Sequences & Series

Theory

Arithmetic Progressions

An *Arithmetic Progression* is a series whose subsequent series is a constant value greater (or less) than the preceding one. Some examples are:

$$1, 2, 3, 4, 5, 6, 7, 8, \ldots \qquad 5, 7, 9, 11, 13, 15, \ldots \qquad 7, 16, 25, 34, 43, 52, \ldots$$

In the general case let the initial term be a and let the difference be d. So the sequence goes:

$$a, a+d, a+2d, a+3d, a+4d, a+5d, a+6d, \ldots, a+(n-1)d$$

So for any sequence the n^{th} term is simply $\boxed{a_n = a+(n-1)d}$. just plug in a, d and n for the desired term.

Summing Arithmetic Progressions

Now we come to the question of summing the first n terms of an arithmetic progression. So we are considering the sum of the following:

$$a, a+d, a+2d, \ldots, a+(n-3)d, a+(n-2)d, a+(n-1)d$$

So let us consider the following sum, S of the progression

$$S = (a) + (a+d) + (a+2d) + \cdots + (a+(n-3)d) + (a+(n-2)d) + (a+(n-1)d)$$

Now to tidy up the algebra we can denote $(a+(n-1)d)$ as l (for last term). So

$$S = (a) + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + (l)$$

Now S can also be thought of as

$$S = (l) + (l-d) + (l-2d) + \cdots + (a+2d) + (a+d) + (a)$$

We can sum (term by term) the two equations for S giving

$$\begin{aligned} 2S = {} & [(a)+(l)] + [(a+d)+(l-d)] + [(a+2d)+(l-2d)] + \cdots \\ & \cdots + [(l-2d)+(a+2d)] + [(l-d)+(a+d)] + [(l)+(a)] \end{aligned}$$

Now we can see that everything in $[]s$ is the same! So

$$2S = [a+l] + [a+l] + [a+l] + \cdots + [a+l] + [a+l] + [a+l]$$

and there are n terms so $2S = n[a+l] \Rightarrow \boxed{S = \frac{n}{2}(a+l)}$ which implies

$$\boxed{S = \frac{n}{2}\left(2a + (n-1)\,d\right)}$$

And that's it! Just plug in the values for a, d and n and the answer will drop out. You might need to rearrange the formula, but that should be trivial.

Geometric Progressions

Whereas an Arithmetic Progression is one where the difference between subsequent terms is a constant ($a_{n+1} - a_n = d = a_n - a_{n-1}$) a *Geometric Progression* is one where the *ratio* between subsequent terms is constant. That is

$$\frac{a_{n+1}}{a_n} = r = \frac{a_n}{a_{n-1}}$$

Some examples are

$$1, 2, 4, 8, 16, 32, 64, \dots \qquad 5, 50, 500, 5000, \dots \qquad 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$$

So if a is the first term, the sequence goes

$$a, ar, ar^2, ar^3, \dots, ar^{n-1}$$

So $\boxed{a_n = ar^{n-1}}$. As always all you need to do to find a term is plug in a, r and n.

Summing Geometric Progressions

As with Arithmetic Progressions we are looking to find the sum of the first n terms.

$$S = a + ar + ar^2 + ar^3 + \dots + ar^{n-2} + ar^{n-1}$$

Multiplying by r gives

$$rS = ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n-1} + ar^n$$

Subtracting one from the other (whilst noticing that most of the terms cancel out) gives $S - rS = a - ar^n$, therefore

$$\boxed{S = a\left(\frac{1 - r^n}{1 - r}\right) = a\left(\frac{r^n - 1}{r - 1}\right)}$$

Now one interesting property that results from this result is if we consider what happens when we let $-1 < r < 1$ and let $n \to \infty$. Obviously if $r > 1$ then as n gets larger, so does S, without bound. But if r is contained within the bound set, then $\lim_{n\to\infty}(r^n) = 0$. So if r is in the range then the *sum to infinity* (S_∞) of the series is

$$\boxed{S_\infty = \frac{a}{1 - r}}$$

What you *need* to know

Calculation	Arithmetic	Geometric
n^{th} term	$a_n = a + (n-1)d$	$a_n = ar^{n-1}$
Sum of n terms	$S = \frac{n}{2}(a + l) = \frac{n}{2}(2a + (n-1)d)$	$S = a\left(\frac{1 - r^n}{1 - r}\right) = a\left(\frac{r^n - 1}{r - 1}\right)$
Sum to infinity	Not possible	$S_\infty = \frac{a}{1 - r}$ if $-1 < r < 1$

Examples

To come.